a) INTEGRACE POMOCÍ ÚPRAVY NA ČTVEREC

b) INTEGRACE POMOCÍ PARCIÁLNÍCH ZLOMKŮ

V ČITATELI MUSÍ BÝT POLYNOM VŽDY ALESPOŇ O JEDEN STUPEŇ NIŽŠÍ NEŽ VE JMENOVATELI

$$\boxed{\int \frac{5x+7}{x^2+4x-5}\,dx}$$

$$\int \frac{5x+7}{x^2+4x-5}\,dx$$

$D = b^2 - 4ac = 4^2 - 4 \cdot 1 \cdot (-5) = 16 + 20 = 36$

$D > 0$ INTEGRACE POMOCÍ PARCIÁLNÍCH ZLOMKŮ → URČÍM KOŘENY

$$\int \frac{5x+7}{x^2+4x-5}\,dx = \underbrace{\int \frac{5x+7}{(x-1)(x+5)}\,dx = \int \left(\frac{A}{x+5} + \frac{B}{x-1}\right) dx}_{} =$$

VE JMENOVATELI JE POLYNOM STUPNĚ DRUHÉHO, MĚL BY MÍT 2 ŘEŠENÍ

KOŘENY:

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x_{1,2} = \frac{-4 \pm \sqrt{4^2 - 4 \cdot 1 \cdot (-5)}}{2 \cdot 1}$$

$$x_{1,2} = \frac{-4 \pm \sqrt{36}}{2}$$

DOSAZUJI DO TVARU $(x - \ldots)$

$$x_{1,2} = \frac{-4 \pm 6}{2} \quad \begin{cases} x_1 = 1 & (x-1) \\ x_2 = -5 & (x-(-5)) \end{cases}$$

$$\frac{5x+7}{(x-1)(x+5)} = \frac{A}{x+5} + \frac{B}{x-1} \quad / \cdot (x+5)(x-1)$$

$$5x + 7 = A(x-1) + B(x+5)$$

$$5x + 7 = Ax - A + Bx + B5$$

$$5x + 7 = Ax - A + Bx + B5$$

$$5x + 7 = (A+B)x + (-A + B5)$$

$$\begin{aligned} A + B &= 5 \\ -A + B5 &= 7 \\ \hline 6B &= 12 \quad /:6 \\ \underline{B} &\underline{= 2} \end{aligned}$$

$A + 2 = 5 \quad /-2$

$\underline{A = 3}$

$$= \int \left(\frac{3}{x+5} + \frac{2}{x-1} \right) dx =$$

$$\int \frac{3}{x+5}\, dx = 3\int \frac{1}{x+5}\, dx = \left| \begin{array}{l} x+5 = s \\ 1\, dx = ds \\ dx = ds \end{array} \right| = 3\int \frac{1}{s}\, ds =$$

$$= 3 \ln |s| + C = 3 \ln |x+5| + C$$

$$\int \frac{2}{x-1}\, dx = 2\int \frac{1}{x-1}\, dx = \left| \begin{array}{l} x-1 = s \\ 1\, dx = ds \\ dx = ds \end{array} \right| = 2\int \frac{1}{s}\, ds =$$

$$= 2 \ln |s| + C = 2 \ln |x-1| + C$$

$$= 3 \ln |x+5| + 2 \ln |x-1| + C =$$

$$= \underline{\underline{\ln |x+5|^3 + \ln |x-1|^2 + C}}$$ (TÉŽ LZE ZAPSAT)

Milan Mroczkowski
yesit.cz
sata150@gmail.com

$$\boxed{\int \frac{x+1}{x^2+x+1}\,dx}$$

$$\int \frac{x+1}{x^2+x+1}\,dx$$

Discriminant jmenovatele je: D = -3, tedy D < 0, tedy nelze určit určit kořeny, tak provedu:

$$\int \frac{x+1}{x^2+x+1}\,dx = \frac{1}{2}\int \frac{2(x+1)}{x^2+x+1}dx = \frac{1}{2}\int \frac{2x+2}{x^2+x+1}dx = \frac{1}{2}\int \frac{2x+1+1}{x^2+x+1}dx =$$

$(x^2+x+1)' = 2x+1$

$\frac{\text{ČITATEL}}{\text{JMENOVATEL}}$ POTŘEBUJI V ČITATELI

$$= \frac{1}{2}\Big(\underbrace{\int \frac{2x+1}{x^2+x+1}\,dx}_{I_1} + \underbrace{\int \frac{1}{x^2+x+1}\,dx}_{I_2}\Big)$$

$$I_1 = \int \frac{2x+1}{x^2+x+1}\,dx = \underline{\ln|x^2+x+1| + C}$$

DLE VZORCE:

$$\int \frac{f'(x)}{f(x)} = \ln|f(x)|$$

$$I_2 = \int \frac{1}{x^2+x+1}\,dx =$$

$x^2+x+1=0$

$D = b^2-4ac = 1^2-4\cdot1\cdot1 = -3$

$D<0$ ÚPRAVA NA ČTVEREC

$D>0$ PARCIÁLNÍ ZLOMEK

$$x^2+x+1$$

$$(x^2+x)+1$$

$$\left[x^2+1x+\left(\frac{1}{2}\right)^2\right] - \left(\frac{1}{2}\right)^2 + 1$$

$$\left(x^2+x+\frac{1}{4}\right) - \frac{1}{4} + 1$$

$$\left(x+\frac{1}{2}\right)^2 + \frac{3}{4}$$

$$\int \frac{1}{x^2+1}\,dx = \operatorname{arctg} x + C$$

$$= \int \frac{1}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}\,dx = \text{"VYTKNU } \frac{3}{4}\text{"} = \int \frac{1}{\frac{3}{4}\left[\frac{\left(x+\frac{1}{2}\right)^2}{\frac{3}{4}}+1\right]}\,dx = \frac{\text{"1. ODMOCNIT"}}{\text{2. ZPĚT UMOCNIT}} =$$

$$= \int \frac{1}{\frac{3}{4}\left[\left(\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)^2+1\right]}\,dx = \int \frac{1}{\frac{3}{4}} \cdot \frac{1}{\left(\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)^2+1}\,dx = \frac{4}{3}\int \frac{1}{\left(\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)^2+1}\,dx =$$

$$= \left| \begin{array}{l} \frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}} = s \\ \frac{2}{\sqrt{3}}\,dx = ds \\ dx = \frac{ds}{\frac{2}{\sqrt{3}}} = \frac{\sqrt{3}}{2}\,ds \end{array} \right| =$$

$$\left[\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right]' = \left(x+\frac{1}{2}\right)' \cdot \frac{1}{\frac{\sqrt{3}}{2}} + \left(x+\frac{1}{2}\right) \cdot \left(\frac{1}{\frac{\sqrt{3}}{2}}\right)' =$$

$$= 1 \cdot \frac{2}{\sqrt{3}} + \left(x+\frac{1}{2}\right) \cdot 0 =$$

$$= \frac{2}{\sqrt{3}}$$

$$= \frac{4}{3}\int \frac{1}{s^2+1} \cdot \frac{\sqrt{3}}{2}\,ds = \frac{4\sqrt{3}}{6}\int \frac{1}{s^2+1}\,ds = \frac{2\sqrt{3}}{3}\int \frac{1}{s^2+1}\,ds =$$

$$= \frac{2\sqrt{3}}{3}\,\mathrm{arctg}\, s + C = \underline{\frac{2\sqrt{3}}{3}\,\mathrm{arctg}\left(\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right) + C}$$

$$= \frac{1}{2}\left[\left(\ln|x^2+x+1| + \frac{2\sqrt{3}}{3}\,\mathrm{arctg}\left(\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)\right] + C$$

$$\boxed{\int \frac{1}{4x^2-9}\,dx} = \int \frac{1}{(2x-3)(2x+3)}\,dx = \int \left(\frac{A}{2x-3} + \frac{B}{2x+3}\right) dx =$$

D>0

$$\frac{1}{(2x-3)(2x+3)} = \frac{A}{2x-3} + \frac{B}{2x+3} \quad / \cdot (2x-3)(2x+3)$$

$$1 = A(2x+3) + B(2x-3)$$

$$1 = 2Ax + 3A + 2Bx - 3B$$

$$0x + 1 = (2A+2B)x + (3A-3B)$$

$$\begin{aligned} 2A + 2B &= 0 \quad /\cdot 3 \\ 3A - 3B &= 1 \quad /\cdot 2 \end{aligned}$$

$$\begin{aligned} 6A + 6B &= 0 \\ 6A - 6B &= 2 \end{aligned} \quad /+$$

$$12A = 2 \quad /:12$$

$$\underline{A} = \frac{2}{12} = \underline{\frac{1}{6}}$$

$$2\frac{1}{6} + 2B = 0$$

$$\frac{1}{3} + 2B = 0 \quad / -\frac{1}{3}$$

$$2B = -\frac{1}{3} \quad /:2$$

$$\underline{B} = -\frac{1}{3} \cdot \frac{1}{2} = \underline{-\frac{1}{6}}$$

$$= \int \left(\frac{\frac{1}{6}}{\underset{\mathrm{I}}{2x-3}} + \frac{-\frac{1}{6}}{\underset{\mathrm{II}}{2x+3}} \right) dx =$$

I $$\int \frac{\frac{1}{6}}{2x-3}\,dx = \left| \begin{aligned} 2x-3 &= t \\ 2\,dx &= dt \\ dx &= \frac{dt}{2} \end{aligned} \right| = \int \frac{\frac{1}{6}}{t}\,\frac{dt}{2} = \frac{\frac{1}{6}}{2} \int \frac{dt}{t} =$$

$$= \frac{1}{12} \int \frac{1}{t}\,dt = \frac{1}{12} \ln|t| + C =$$

$$= \frac{1}{12} \ln|2x-3| + C$$

II $$\int \frac{-\frac{1}{6}}{2x+3}\,dx = \left| \begin{aligned} 2x+3 &= t \\ 2\,dx &= dt \\ dx &= \frac{dt}{2} \end{aligned} \right| = \int \frac{-\frac{1}{6}}{t}\,\frac{dt}{2} = \frac{-\frac{1}{6}}{2} \int \frac{dt}{t} =$$

$$= -\frac{1}{12} \int \frac{1}{t}\,dt = -\frac{1}{12} \ln|t| + C =$$

$$= -\frac{1}{12} \ln|2x+3| + C$$

$$= \underline{\underline{\frac{1}{12} \ln|2x-3| - \frac{1}{12} \ln|2x+3| + C}}$$

$$\boxed{\int \frac{3x^2+3x+5}{x^2+2x-3}\,dx}$$

$$\int \frac{3x^2+3x+5}{x^2+2x-3}\,dx$$

$$\begin{array}{l} (3x^2+3x+5) : (x^2+2x-3) = 3 + \dfrac{-3x+14}{x^2+2x-3} \\ -\underline{(3x^2+6x-9)} \\ \quad -3x+14 \end{array}$$

$$\int 3 + \frac{-3x+14}{x^2+2x-3}\,dx = \underbrace{3\int 1\,dx}_{\mathrm{I}} + \underbrace{\int \frac{-3x+14}{x^2+2x-3}\,dx}_{\mathrm{II}}$$

$$\mathrm{I} = 3\int 1\,dx = 3x + C$$

$$\mathrm{II} = \int \frac{-3x+14}{x^2+2x-3}\,dx =$$

$D > 0$

$$x_{1,2} = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$x_{1,2} = \frac{-2 \pm \sqrt{2^2-4\cdot 1\cdot(-3)}}{2\cdot 1}$$

$$x_{1,2} = \frac{-2 \pm 4}{2} \begin{cases} 1 = x_1 & (x-1) \\ -3 = x_2 & (x-(-3)) \end{cases}$$

$$= \int \frac{-3x+14}{(x-1)(x+3)}\,dx = \int \left(\frac{A}{(x-1)} + \frac{B}{(x+3)}\right) dx =$$

$$\frac{-3x+14}{(x-1)(x+3)} = \frac{A}{x-1} + \frac{B}{x+3} \quad \Big/ \cdot (x-1)(x+3)$$

$$-3x+14 = A(x+3) + B(x-1)$$

$$-3x+14 = Ax + 3A + Bx - B$$

$$-3x+14 = (A+B)x + (3A-B)$$

$$A + B = -3$$
$$3A - B = 14 \quad /+$$
$$4A = 11 \quad /:4$$
$$A = \frac{11}{4}$$

$$\frac{11}{4} + B = -3 \quad / -\frac{11}{4}$$
$$B = -3 - \frac{11}{4}$$
$$B = -\frac{3}{1} - \frac{11}{4} = \frac{-12-11}{4} = -\frac{23}{4}$$

$$= \int \left(\frac{\frac{11}{4}}{x-1} + \frac{-\frac{23}{4}}{x+3} \right) dx = \underbrace{\frac{11}{4} \int \frac{1}{x-1}\, dx}_{I_1} + \underbrace{\left(-\frac{23}{4}\right) \int \frac{1}{x+3}\, dx}_{I_2}$$

$$I_1 = \frac{11}{4} \int \frac{1}{x-1}\, dx = \left| \begin{array}{l} x-1 = t \\ 1\,dx = dt \\ dx = dt \end{array} \right| = \frac{11}{4} \int \frac{1}{t}\, dt = \frac{11}{4} \ln|t| + C =$$
$$= \frac{11}{4} \ln|x-1| + C$$

$$I_2 = -\frac{23}{4} \int \frac{1}{x+3}\, dx = \left| \begin{array}{l} x+3 = t \\ 1\,dx = dt \\ dx = dt \end{array} \right| = -\frac{23}{4} \int \frac{1}{t}\, dt = -\frac{23}{4} \ln|t| + C =$$
$$= -\frac{23}{4} \ln|x+3| + C$$

$$\underline{\underline{3x + \frac{11}{4} \ln|x-1| - \frac{23}{4} \ln|x+3| + C}}$$

$$\boxed{\int \frac{x^2-x+4}{x^2+2x+1}\,dx}$$

$$\int \frac{x^2-x+4}{x^2+2x+1}\,dx =$$

$$(x^2-x+4):(x^2+2x+1) = 1 + \frac{-3x+3}{x^2+2x+1}$$
$$\underline{-(x^2+2x+1)}$$
$$-3x+3$$

$$= \int 1 + \frac{-3x+3}{x^2+2x+1}\,dx = \underbrace{\int 1\,dx}_{\mathrm{I}} + \underbrace{\int \frac{-3x+3}{x^2+2x+1}\,dx}_{\mathrm{II}}$$

$$\mathrm{I} = \int 1\,dx = \underline{x+C}$$

$$\mathrm{II} = \int \frac{-3x+3}{x^2+2x+1}\,dx = \int\left(\frac{A}{x+1} + \frac{B}{(x+1)^2}\right)dx$$

$D = 0$

$$x_{1,2} = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$x_{1,2} = \frac{-2 \pm \sqrt{2^2-4\cdot 1\cdot 1}}{2\cdot 1}$$

$$x_{1,2} = \frac{-2 \pm 0}{2} \quad \begin{cases} -1 & (x-(-1)) \\ -1 & (x-(-1)) \end{cases}$$

$$\frac{-3x+3}{(x+1)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2} \quad /\cdot (x+1)^2$$

$$-3x+3 = A(x+1) + B$$

$$-3x+3 = Ax + A + B$$

$$-3x+3 = (A)x + (A+B)$$

$$A = -3$$
$$\underline{A + B = 3}$$

$$\underline{A = -3} \qquad -3 + B = 3 \quad /+3$$
$$\underline{B = 6}$$

$$\int \left(\frac{-3}{x+1} + \frac{6}{(x+1)^2} \right) dx = -3\int \frac{1}{x+1}\,dx + 6\int \frac{1}{(x+1)^2}\,dx$$

II_1 II_2

II_1

$$-3\int \frac{1}{x+1}\,dx = \left| \begin{array}{l} x+1 = s \\ 1\,dx = ds \\ dx = ds \end{array} \right| = -3\int \frac{1}{s}\,ds = -3\ln|s| + C =$$

$$= \underline{-3\ln|x+1| + C}$$

II_2

$$6\int \frac{1}{(x+1)^2}\,dx = \left| \begin{array}{l} x+1 = s \\ 1\,dx = ds \\ dx = ds \end{array} \right| = 6\int \frac{1}{s^2}\,ds = 6\int s^{-2}\,ds = 6 \cdot \frac{s^{-2+1}}{-2+1} + C =$$

$$= 6 \cdot \frac{s^{-1}}{-1} + C =$$

$$= 6 \cdot \frac{\frac{1}{s}}{-1} + C =$$

$$= 6 \cdot \left(-\frac{1}{s}\right) + C =$$

$$= \underline{-\frac{6}{x+1} + C}$$

$$\underline{\underline{x + (-3\ln|x+1|) + \left(-\frac{6}{x+1}\right) + C}}$$

MILAN MROCZKOWSKI SATA150@GMAIL.COM YESIT.CZ

$$\boxed{\int \frac{1}{x^2(x-1)^3}\,dx}$$

JE TO POLYNOM 5. STUPNĚ, TAKŽE BYCH MĚL MÍT 5 ŘEŠENÍ

$$x^2(x-1)^3 = x^2(x^3-3x^2+3x-1) = x^5-3x^4+3x^3-x^2$$

5. STUPNĚ

$$\int \frac{1}{x^2(x-1)^3}\,dx = \int\left(\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x-1}+\frac{D}{(x-1)^2}+\frac{E}{(x-1)^3}\right)dx$$

$$\frac{1}{x^2(x-1)^3} = \frac{A}{x}+\frac{B}{x^2}+\frac{C}{x-1}+\frac{D}{(x-1)^2}+\frac{E}{(x-1)^3} \quad /\cdot[x^2(x-1)^3]$$

$$1 = A[x(x-1)^3]+B(x-1)^3+C[x^2(x-1)^2]+D[x^2(x-1)]+Ex^2$$

$$1 = A[x(x^3-3x^2+3x-1)]+B[x^3-3x^2+3x-1]+C[x^2(x^2-2x+1)]+D(x^3-x^2)+Ex^2$$

$$1 = A[x^4-3x^3+3x^2-x]+B[x^3-3x^2+3x-1]+C[x^4-2x^3+x^2]+D(x^3-x^2)+Ex^2$$

$$1 = Ax^4-3Ax^3+A3x^2-Ax+Bx^3-B3x^2+B3x-B+Cx^4-C2x^3+Cx^2+Dx^3-Dx^2+Ex^2$$

$$1 = (A+C)x^4+(-A3+B-C2+D)x^3+(A3-B3+C-D+E)x^2+(-A+B3)x-B$$

$$A+C=0$$
$$-A3+B-C2+D=0$$
$$A3-B3+C-D+E=0$$
$$-A+B3=0$$
$$-B=1$$

$-B=1$

$\underline{B=-1}$

$-A+(-1)\cdot 3=0$

$-A-3=0 \quad /+3$

$-A=3 \quad /\cdot(-1)$

$\underline{A=-3}$

$-3+C=0$

$\underline{C=3}$

$-(-3)\cdot 3+(-1)-3\cdot 2+D=0$

$9+(-1)-6+D=0$

$2+D=0$

$\underline{D=-2}$

$-3\cdot 3-(-1)\cdot 3+3-(-2)+E=0$

$-9-(-3)+5+E=0$

$-6+5+E=0$

$-1+E=0 \quad /+1$

$\underline{E=1}$

$$\int\left(\frac{-3}{x}+\frac{-1}{x^2}+\frac{3}{x-1}+\frac{-2}{(x-1)^2}+\frac{1}{(x-1)^3}\right)dx$$

$$\qquad \mathrm{I} \qquad \mathrm{II} \qquad \mathrm{III} \qquad \mathrm{IV} \qquad \mathrm{V}$$

I $\quad -3\int\frac{1}{x}\,dx = \underline{-3\ln|x|+C}$

II $\quad -1\int\frac{1}{x^2}\,dx = -1\int x^{-2}\,dx = -1\cdot\frac{x^{-1}}{-1}+C = -1\cdot\frac{\frac{1}{x}}{-1}+C = \underline{\frac{1}{x}+C}$

III $\quad 3\int\frac{1}{x-1}\,dx = \left|\begin{matrix} x-1=s \\ 1\,dx=ds \\ dx=ds\end{matrix}\right| = 3\int\frac{1}{s}\,ds = 3\ln|s|+C =$

$$= \underline{3\ln|x-1|+C}$$

IV $\quad -2\int\frac{1}{(x-1)^2}\,dx = \left|\begin{matrix} x-1=s \\ 1\,dx=ds \\ dx=ds\end{matrix}\right| = -2\int\frac{1}{s^2}\,ds = -2\int s^{-2}\,ds = -2\cdot\frac{s^{-1}}{-1}+C =$

$$= -2\,\frac{\frac{1}{s}}{-1}+C = \frac{2}{s}+C = \underline{\frac{2}{x-1}+C}$$

V $\quad \int\frac{1}{(x-1)^3}\,dx = \left|\begin{matrix} x-1=s \\ 1\,dx=ds \\ dx=ds\end{matrix}\right| = \int\frac{1}{s^3}\,ds = \int s^{-3}\,ds = \frac{s^{-2}}{-2}+C =$

$$= \frac{\frac{1}{s^2}}{-2}+C = \frac{1}{-2s^2}+C = \underline{\frac{1}{-2(x-1)^2}+C}$$

$$\underline{\underline{-3\ln|x|+\frac{1}{x}+3\ln|x-1|+\frac{2}{x-1}+\frac{1}{-2(x-1)^2}+C}}$$

MILAN
MROCZKOWSKI YESIT.CZ SATA150@GMAIL.COM

$$\boxed{\int \frac{1}{x^3-x^2}\,dx}$$

$$\int \frac{1}{x^2(x-1)}\,dx = \int\left(\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x-1}\right)dx$$

$$\frac{1}{x^2(x-1)} = \frac{A}{x}+\frac{B}{x^2}+\frac{C}{x-1} \quad /\cdot x^2(x-1)$$

$$1 = A[x(x-1)] + B(x-1) + Cx^2$$

$$1 = A(x^2-x) + Bx - B + Cx^2$$

$$1 = Ax^2 - Ax + Bx - B + Cx^2$$

$$0x^2 + 0x + 1 = (A+C)x^2 + (-A+B)x - B$$

$$\begin{aligned} A + C &= 0 \\ -A + B &= 0 \\ -B &= 1 \end{aligned}$$

$$-B = 1 \quad /\cdot(-1)$$
$$\underline{B = -1}$$

$$-A + (-1) = 0$$
$$-A - 1 = 0 \quad /+1$$
$$-A = 1 \quad /\cdot(-1)$$
$$\underline{A = -1}$$

$$(-1) + C = 0$$
$$\underline{C = 1}$$

$$\int\left(\underbrace{\frac{-1}{x}}_{\text{I}} + \underbrace{\frac{-1}{x^2}}_{\text{II}} + \underbrace{\frac{1}{x-1}}_{\text{III}}\right)dx$$

$$\text{I} \quad -1\int \frac{1}{x}\,dx = \underline{-\ln|x| + C}$$

$$\text{II} \quad -1\int \frac{1}{x^2}\,dx = -1\int x^{-2}\,dx = -1\cdot\frac{x^{-1}}{-1} + C = -1\cdot\frac{\frac{1}{x}}{-1} + C = \underline{\frac{1}{x} + C}$$

$$\text{III} \quad \int \frac{1}{x-1}\,dx = \left|\begin{matrix} x-1 = t \\ 1dx = dt \\ dx = dt \end{matrix}\right| = \int \frac{1}{t}\,dt = \ln|t| + C = \underline{\ln|x-1| + C}$$

$$\underline{\underline{-\ln|x| + \frac{1}{x} + \ln|x-1| + C}}$$

STEJNÝ PŘÍKLAD, ALE JINÝ POSTUP

$$\boxed{\int \frac{1}{x^3-x^2}\,dx}$$

$$\int \frac{1}{x^2(x-1)}\,dx = \int\left(\frac{Ax+C}{x^2} + \frac{B}{x-1}\right)dx$$

POZNÁMKA:

$$\frac{Ax}{x^2} + \frac{C}{x^2} = \frac{A}{x} + \frac{C}{x^2}$$

$$\frac{1}{x^2(x-1)} = \frac{Ax+C}{x^2} + \frac{B}{x-1} \quad / \cdot x^2(x-1)$$

$$1 = (Ax+C)(x-1) + Bx^2$$

$$1 = Ax^2 - Ax + Cx - C + Bx^2$$

$$0x^2 + 0x + 1 = (A+B)x^2 + (-A+C)x - C$$

$$A + B = 0$$
$$-A + C = 0$$
$$-C = 1$$

$$-C = 1 \quad /\cdot(-1)$$
$$\underline{C = -1}$$

$$-A + (-1) = 0$$
$$-A = 1 \quad /\cdot(-1)$$
$$\underline{A = -1}$$

$$-1 + B = 0 \quad /+1$$
$$\underline{B = 1}$$

$$\int\left(\frac{-1x+(-1)}{x^2} + \frac{1}{x-1}\right)dx = \int\left(-\frac{x}{x^2} - \frac{1}{x^2} + \frac{1}{x-1}\right)dx =$$

$$= \int\left(\underbrace{-\frac{1}{x}}_{\mathrm{I}} \underbrace{- \frac{1}{x^2}}_{\mathrm{II}} + \underbrace{\frac{1}{x-1}}_{\mathrm{III}}\right)dx$$

DÁL JE TO STEJNÝ JAKO U PŘEDCHOZÍHO POSTUPU

$$\underline{\underline{-\ln|x| + \frac{1}{x} + \ln|x-1| + C}}$$

POZNÁMKA:

$$(a^3-b^3) = (a-b)(a^2+ab+b^2)$$

$$(a^3+b^3) = (a+b)(a^2-ab+b^2)$$

$$(a-b)^3 = a^3-3a^2b+3ab^2-b^3$$

$$(a+b)^3 = a^3+3a^2b+3ab^2+b^3$$

$$\boxed{\int \frac{1}{x^3-1}\,dx}$$

POKUD MÁM POLYNOM VE JMENOVATELI VYŠŠÍHO STUPNĚ NEŽ 2, TAK VŽDY POUŽIJ INTEGRACI POMOCÍ PARCIÁLNÍCH ZLOMKŮ.

$$\int \frac{1}{x^3-1}\,dx = \int \frac{1}{(x-1)(x^2+x+1^2)}\,dx = \int \left(\frac{A}{x-1} + \frac{Bx+C}{x^2+x+1} \right) dx =$$

$$\frac{1}{(x-1)(x^2+x+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1} \quad / \cdot (x-1)(x^2+x+1)$$

$$1 = A(x^2+x+1) + (Bx+C)(x-1)$$

$$1 = Ax^2 + Ax + A + Bx^2 - Bx + Cx - C$$

$$0x^2 + 0x + 1 = (A+B)x^2 + (A-B+C)x + (A-C)$$

$$A + B = 0$$
$$A - B + C = 0$$
$$A - C = 1$$

$$\begin{aligned} A + B \quad &= 0 \\ A - B + C &= 0 \\ A \quad - C &= 1 \end{aligned} \quad \Big/ +$$

$$3A = 1$$
$$\underline{A = \frac{1}{3}}$$

$$\frac{1}{3} + B = 0$$
$$\underline{B = -\frac{1}{3}}$$

$$\frac{1}{3} - C = 1$$
$$-C = \frac{1}{1} - \frac{1}{3}$$
$$-C = \frac{3-1}{3}$$
$$-C = \frac{2}{3}$$
$$\underline{C = -\frac{2}{3}}$$

$$= \int \left(\frac{\frac{1}{3}}{x-1} + \frac{-\frac{1}{3}x - \frac{2}{3}}{x^2+x+1} \right) dx = \int \left(\frac{1}{3}\frac{1}{x-1} + \left(-\frac{1}{3}\right)\frac{x+2}{x^2+x+1} \right) dx =$$

$$= \frac{1}{3}\int \frac{1}{x-1}\,dx + \left(-\frac{1}{3}\right)\int \frac{x+2}{x^2+x+1}\,dx$$

I $\qquad$ II

I $\quad \frac{1}{3}\int \frac{1}{x-1}\,dx = \underline{\frac{1}{3}\ln|x-1|+C}$

VZOREC

$$\boxed{\int \frac{f'(x)}{f(x)}\,dx = \ln|x|+C}$$

jmenovatel $(x-1)' = 1$ čitateli

II $\quad -\frac{1}{3}\int \frac{x+2}{x^2+x+1}\,dx = -\frac{1}{3}\cdot\frac{1}{2}\int \frac{2(x+2)}{x^2+x+1}\,dx = -\frac{1}{6}\int \frac{2x+4}{x^2+x+1}\,dx =$

$$= -\frac{1}{6}\int \frac{2x+1+3}{x^2+x+1}\,dx = -\frac{1}{6}\int \left(\underbrace{\frac{2x+1}{x^2+x+1}}_{\mathrm{II}_1} + \underbrace{\frac{3}{x^2+x+1}}_{\mathrm{II}_2}\right)dx$$

II_1 $\quad -\frac{1}{6}\int \frac{2x+1}{x^2+x+1}\,dx = \underline{-\frac{1}{6}\ln|x^2+x+1|+C}$

II_2 $\quad -\frac{1}{6}\cdot 3\int \frac{1}{x^2+x+1}\,dx = -\frac{1}{2}\int \frac{1}{x^2+x+1} =$

$D<0$

$(x^2+x)+1$

$\left[x^2+x+\left(\frac{1}{2}\right)^2\right]-\left(\frac{1}{2}\right)^2+\frac{1}{1}$

$\left(x^2+x+\frac{1}{4}\right)-\frac{1}{4}+\frac{1}{1}$

$\left(x+\frac{1}{2}\right)^2+\frac{3}{4}$

$$= -\frac{1}{2}\int \frac{1}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}\,dx = -\frac{1}{2}\int \frac{1}{\frac{3}{4}\left[\frac{\left(x+\frac{1}{2}\right)^2}{\frac{3}{4}}+1\right]}\,dx = -\frac{1}{2}\cdot\frac{4}{3}\int \frac{1}{\left(\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)^2+1}\,dx =$$

$$= -\frac{4}{6}\int \frac{1}{\left(\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)^2+1} = \left| \begin{array}{l} \dfrac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}} = t \\ \dfrac{2}{\sqrt{3}}\,dx = dt \\ dx = \dfrac{\sqrt{3}}{2}\,dt \end{array} \right| = -\frac{4}{6}\int \frac{1}{t^2+1}\cdot\frac{\sqrt{3}}{2}\,dt =$$

$$\left(\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)' = \left[\left(x+\frac{1}{2}\right)\cdot\frac{1}{\frac{\sqrt{3}}{2}}\right]' = \left(x+\frac{1}{2}\right)'\cdot\frac{2}{\sqrt{3}} + \left(x+\frac{1}{2}\right)\cdot\left(\frac{2}{\sqrt{3}}\right)' =$$

$$= 1\cdot\frac{2}{\sqrt{3}} + \left(x+\frac{1}{2}\right)\cdot 0 = \frac{2}{\sqrt{3}}$$

$$= -\frac{2}{3}\cdot\frac{\sqrt{3}}{2}\int\frac{1}{t^2+1}\,dt = -\frac{\sqrt{3}}{3}\int\frac{1}{t^2+1}\,dt = -\frac{\sqrt{3}}{3}\,\mathrm{arctg}\,(t) + C =$$

$$= -\frac{\sqrt{3}}{3}\,\mathrm{arctg}\left(\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right) + C =$$

$$= -\frac{\sqrt{3}}{3}\,\mathrm{arctg}\left(\frac{2x+1}{\sqrt{3}}\right) + C$$

$$\frac{1}{3}\ln|x-1| - \frac{1}{6}\ln|x^2+x+1| - \frac{\sqrt{3}}{3}\,\mathrm{arctg}\left(\frac{2x+1}{\sqrt{3}}\right) + C$$

$$\boxed{\int \frac{e^x-1}{e^x+1}\,dx} = \left| \begin{aligned} e^x &= s \\ e^x dx &= ds \\ dx &= \frac{ds}{e^x} = \frac{ds}{s} = \frac{1}{s}ds \end{aligned} \right| = \int \frac{s-1}{s+1}\cdot\frac{1}{s}\,ds =$$

$$= \int \frac{s-1}{s(s+1)}\,ds = \int\left(\frac{A}{s} + \frac{B}{s+1}\right)ds =$$

$$\frac{s-1}{s(s+1)} = \frac{A}{s} + \frac{B}{s+1} \qquad / \cdot s(s+1)$$

$$s-1 = A(s+1) + Bs$$

$$s-1 = As + A + Bs$$

$$s-1 = (A+B)s + (A)$$

$$\begin{aligned} A + B &= 1 \\ A &= -1 \end{aligned}$$

$\underline{A=-1}$

$$-1 + B = 1 \quad /+1$$

$$\underline{B = 2}$$

$$= \int\left(\underset{\mathrm{I}}{\frac{-1}{s}} + \underset{\mathrm{II}}{\frac{2}{s+1}}\right)ds$$

$$\mathrm{I} \quad \int \frac{-1}{s}\,ds = -1\int\frac{1}{s}\,ds = -\ln|s| + C = \underline{-\ln|e^x| + C}$$

$$\mathrm{II} \quad \int \frac{2}{s+1}\,ds = 2\int\frac{1}{s+1}\,ds = 2\ln|s+1| + C = \underline{2\ln|e^x+1| + C}$$

$$\underline{\underline{-\ln|e^x| + 2\ln|e^x+1| + C}}$$

METODA PER-PARTES NEURČITÉHO INTEGRÁLU

(INTEGRACE PO ČÁSTECH)

$$\boxed{\int u \cdot v' \, dx = u \cdot v - \int u' \cdot v \, dx}$$

1. KROK 2. KROK

$$\int x \cdot e^x \, dx = \left| \begin{array}{ll} u = x & u' = 1 \\ v' = e^x & v = e^x \end{array} \right| =$$

• 3KROK 4.

CO BUDU CHTÍT DERIVOVAT? X

$$= x \cdot e^x - \int 1 \cdot e^x \, dx =$$

$$= \underline{\underline{x \cdot e^x - e^x + C}}$$

DERIVUJI

$$\int x \cdot \ln x \, dx = \int \ln x \cdot x \, dx = \left| \begin{array}{ll} u = \ln x & u' = \frac{1}{x} \\ v' = x & v = \frac{x^2}{2} \end{array} \right| =$$

INTEGRUJI

CO BUDU CHTÍT DERIVOVAT? $\ln x$

$$= \ln x \cdot \frac{x^2}{2} - \int \frac{1}{x} \cdot \frac{x^2}{2} \, dx =$$

$$= \ln x \cdot \frac{x^2}{2} - \frac{1}{2} \int \frac{x^2}{x} \, dx =$$

$$= \ln x \cdot \frac{x^2}{2} - \frac{1}{2} \int x \, dx =$$

$$= \underline{\underline{\ln x \cdot \frac{x^2}{2} - \frac{1}{2} \frac{x^2}{2} + C}}$$

CO BUDU CHTÍT DERIVOVAT? $\ln x$

$$\int \ln x \, dx = \int 1 \cdot \ln x \, dx = \int \ln x \cdot 1 \, dx =$$

$$= \left| \begin{array}{ll} u = \ln x & u' = \frac{1}{x} \\ v' = 1 & v = x \end{array} \right| = \ln x \cdot x - \int \frac{1}{x} \cdot x \, dx =$$

$$= \underline{\underline{\ln x \cdot x - x + C}}$$

$$\int \operatorname{arctg} x\,dx = \int \operatorname{arctg} x \cdot 1\,dx = \left| \begin{array}{ll} u = \operatorname{arctg} x & u' = \dfrac{1}{1+x^2} \\ v' = 1 & v = x \end{array} \right| =$$

$$= \operatorname{arctg} x \cdot x - \int \frac{1}{1+x^2} \cdot x\,dx =$$

$$= \operatorname{arctg} x \cdot x - \left| \begin{array}{r} 1+x^2 = t \\ 2x\,dx = dt \\ dx = \dfrac{dt}{2x} \end{array} \right| =$$

$$= \operatorname{arctg} x \cdot x - \int \frac{1}{t} \cdot x \cdot \frac{dt}{2x} =$$

$$= x \cdot \operatorname{arctg} x - \int \frac{1}{t} \cdot x \cdot \frac{1}{2x}\,dt =$$

$$= x \cdot \operatorname{arctg} x - \int \frac{1}{t} \cdot \frac{x}{2x}\,dt =$$

$$= x \cdot \operatorname{arctg} x - \int \frac{1}{t} \cdot \frac{1}{2}\,dt =$$

$$= x \cdot \operatorname{arctg} x - \frac{1}{2} \int \frac{1}{t}\,dt =$$

$$= x \cdot \operatorname{arctg} x - \frac{1}{2} \ln |t| + C =$$

$$= x \cdot \operatorname{arctg} x - \frac{1}{2} \ln |1+x^2| + C$$

$$\int \sin^2 x\,dx = \int \sin x \cdot \sin x\,dx = \begin{vmatrix} u = \sin x & u' = \cos x \\ v' = \sin x & v = -\cos x \end{vmatrix} =$$

$$= \sin x \cdot (-\cos x) - \int \cos x \cdot (-\cos x)\,dx =$$

$$= \sin x \cdot (-\cos x) - \int -\cos^2 x\,dx =$$

$$= \sin x \cdot (-\cos x) - (-1)\int \cos^2 x\,dx =$$

$$= \sin x \cdot (-\cos x) + \int \cos^2 x\,dx =$$

$$= \sin x \cdot (-\cos x) + \int 1 - \sin^2 x\,dx =$$

$$= \sin x \cdot (-\cos x) + \int 1\,dx - \int \sin^2 x\,dx =$$

$$= \sin x \cdot (-\cos x) + x - \int \sin^2 x\,dx$$

ZMĚNA O PROTI ZADÁNÍ

opiši zadání

$$\int \sin^2 x\,dx = \sin x \cdot (-\cos x) + x - \int \sin^2 x\,dx \quad \Big/ + \int \sin^2 x\,dx$$

$$2\int \sin^2 x\,dx = \sin x \cdot (-\cos x) + x \quad \Big/ : 2$$

Teď s tím pravuji jako s rovnicí, neznámé (integrál) dám na jednu stranu, známé dám na druhou stranu.

$$\int \sin^2 x\,dx = \frac{\sin x \cdot (-\cos x) + x}{2}$$

SUBSTITUCE - OPAKOVÁNÍ

$$\int \frac{1}{(2x+1)^3}\,dx = \left| \begin{array}{l} 2x+1 = s \\ 2\,dx = ds \\ dx = \frac{1}{2}\,ds \end{array} \right| = \int \frac{1}{s^3} \cdot \frac{1}{2}\,ds =$$

$$= \frac{1}{2} \int \frac{1}{s^3}\,ds = \frac{1}{2} \int s^{-3}\,ds =$$

VZOREC $\int x^a \, \cancel{dx} = \frac{x^{a+1}}{a+1} \quad a \neq -1$

$$= \frac{1}{2}\,\frac{s^{-2}}{-2} + C = \frac{1}{2}\,\frac{(2x+1)^{-2}}{-2} + C =$$

$$= \frac{1}{2} \cdot \frac{\frac{1}{(2x+1)^2}}{-2} + C = \frac{1}{2} \cdot \frac{1}{-2(2x+1)^2} + C =$$

$$= \frac{1}{-4(2x+1)^2} + C$$