INTEGRÁL

- OPERACE KTERÁ JDE PROTI DERIVOVÁNÍ

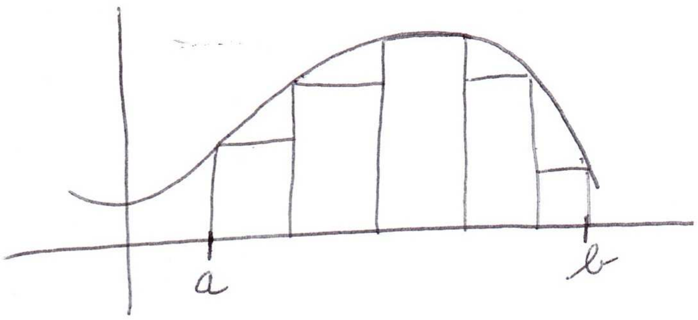
PRIMITIVNÍ FUNKCE

Mám nějakou funkci a chci spočítat integrál:

$$\int_a^b f(x)\,dx$$

REAMANŮV INTEGRÁL

Mám nějakou funkci, co určuji když integruji?



- POČÍTÁM PLOCHU POD KŘIVKOU
- ROZDĚLÍM PLOCHU NA INTERVALY, POKUD JE URČITÝ INTEGRÁL **BUDE INTERVAL** < >, NEURČITÝ INTEGRÁL ( ).

- Problém tohoto přístupu je, že neukazuje jak integrál spočítat, protože to určitým způsobem odhaduji. Pak tam potřebuji limitu a další věci, proto tento přístup není úplně dobrý.

NEWTON - LEIBNIZOVA FORMULE

SLOUŽÍ:
- ABYCHOM MOHLI ŘEŠIT INTEGRÁL

$$\int_a^b f(x)dx = F(b) - F(a)$$

Body: b, a jsou meze

Funkce F je funkce primitivní.

JAKÝ JE VZTAH MEZI f a F?

$$F' = f$$

VZORCE:

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C \qquad n \neq -1$$

ZPĚT: $\dfrac{(x^{n+1})'}{n+1} + (C)' = \dfrac{(n+1)\,x^n}{(n+1)} + 0 = x^n$

INTEGRÁL KONSTANTY

$$\int C\,dx = Cx + K$$

NĚJAKÁ JINÁ KONSTANTA

$$\int 2\,dx = 2x + C$$

v absolutní hodnotě

$$\int \frac{1}{x}\,dx = \ln|x| + C$$

PLATÍ, POUZE KDYŽ x JE KLADNÉ

$$\int e^x\,dx = e^x + C$$

$$\int a^x\,dx = \frac{a^x}{\ln a} + C$$

ZPĚT: $\frac{(a^x)'}{\ln a} = \frac{a^x \ln a}{\ln a} = a^x$

$$\int \sin x\,dx = -\cos x + C$$

$$\int \cos x\,dx = \sin x + C$$

$$\int \frac{1}{\cos^2 x}\,dx = \operatorname{tg} x + C$$

$$\int \frac{1}{\sin^2 x}\,dx = -\operatorname{cotg} x + C$$

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + C$$

$$\int \frac{-1}{\sqrt{1-x^2}}\,dx = \arccos x + C$$

$$\int \frac{1}{1+x^2}\,dx = \operatorname{arctg} x + C$$

nebo: $-\operatorname{arccotg} x + C$

POZNÁMKA:

V DERIVACE DOSTÁVÁM 1 VÝSLEDEK

U INTEGRÁLU DOSTÁVÁM NEKONEČNĚ MNOHO VÝSLEDKŮ -

- NEKONEČNĚ MNOHO FUNKCÍ

Konstanta C ovlivňuje posunutí (nahoru, dolů) po ose y.

$$\int \frac{-1}{1+x^2}\,dx = -(-\operatorname{arccotg} x) + C$$

PŘÍKLADY:

ŘÍKÁ PODLE JAKÉ PROMĚNNÉ MÁME INTEGROVAT

$$\int x^7 - \frac{3}{2}x + 5\,dx = \frac{x^{7+1}}{7+1} + C - \frac{3}{2}\frac{x^2}{1+1} + C + 5x + C =$$

PŘEDSTAVA: $5x^0$

$$\left(\int x^7 dx - \int \frac{3}{2}x\,dx + \int 5\,dx\right)$$

KDYŽ SEČTU KONSTANTU, DOSTANU ZASE KONSTANTU, KTEROU PÍŠI NAKONEC.

$$= \frac{x^8}{8} - \frac{3}{2}\frac{x^2}{2} + 5x + C$$

$$\int \underbrace{x^7 - \frac{3}{2}x + 5}\,dy = x^7 y - \frac{3}{2}xy + 5y + C =$$

VŠE CHÁPU JAKO KONSTANTU, PROTOŽE y TADY NENÍ

$$= \left(x^7 - \frac{3}{2}x + 5\right)y + C$$

$$\int \frac{5}{x} + \frac{3}{x^2} - \frac{x}{4}\,dx = \int 5x^{-1} + 3x^{-2} - \frac{x}{4}\,dx =$$

$$= \int 5\frac{1}{x} + 3\frac{1}{x^2} - \frac{x}{4}\,dx =$$

$$= 5\ln x + 3\cdot\frac{x^{-2+1}}{-2+1} - \frac{1}{4}\frac{x^{1+1}}{1+1} + C =$$

$$= 5\ln x + 3\frac{x^{-1}}{-1} - \frac{1}{4}\frac{x^2}{2} + C$$

$$\int \frac{\sqrt{x} - 2\sqrt[3]{x^2} + 1}{\sqrt[4]{x}} dx = \int \frac{x^{\frac{1}{2}} - 2x^{\frac{2}{3}} + 1}{x^{\frac{1}{4}}} dx =$$

$$= \int \left(\frac{x^{\frac{1}{2}}}{x^{\frac{1}{4}}} - \frac{2x^{\frac{2}{3}}}{x^{\frac{1}{4}}} + \frac{1}{x^{\frac{1}{4}}} \right) dx =$$

$$\frac{2}{3} - \frac{1}{4} = \frac{8-3}{12} = \frac{5}{12}$$

$$\frac{1}{4} + \frac{1}{1} = \frac{1+4}{4} = \frac{5}{4}$$

$$\frac{5}{12} + \frac{1}{1} = \frac{5+12}{12} = \frac{17}{12}$$

$$= \int \left(x^{\frac{1}{4}} - 2x^{\frac{5}{12}} + x^{-\frac{1}{4}} \right) dx =$$

$$= \int x^{\frac{1}{4}} dx - 2\int x^{\frac{5}{12}} dx + \int x^{-\frac{1}{4}} dx =$$

$$= \frac{x^{\frac{1}{4}+1}}{\frac{1}{4}+1} - 2\frac{x^{\frac{5}{12}+1}}{\frac{5}{12}+1} + \frac{x^{-\frac{1}{4}+1}}{-\frac{1}{4}+1} + C =$$

$$= \frac{x^{\frac{5}{4}}}{\frac{5}{4}} - \frac{2x^{\frac{17}{12}}}{\frac{17}{12}} + \frac{x^{\frac{3}{4}}}{\frac{3}{4}} + C =$$

$$= \frac{x^{\frac{5}{4}}}{1} \cdot \frac{4}{5} - \frac{2x^{\frac{17}{12}}}{1} \cdot \frac{12}{17} + \frac{x^{\frac{3}{4}}}{1} \cdot \frac{4}{3} + C =$$

$$= \underline{\underline{\frac{4x^{\frac{5}{4}}}{5} - \frac{24x^{\frac{17}{12}}}{17} + \frac{4x^{\frac{3}{4}}}{3} + C}}$$

$$\int \frac{x^3 + x^2 - 2x}{x-1} dx = \int \frac{x(x^2+x-2)}{x-1} dx =$$

$$= \int \frac{x(x-1)(x+2)}{x-1} dx = \int x(x+2)\, dx =$$

$$= \int x^2 + 2x \, dx = \underline{\underline{\frac{x^3}{3} + 2\frac{x^2}{2} + C}}$$

$$x^2 + x - 2 = 0$$

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x_{1,2} = \frac{-1 \pm \sqrt{1^2 - 4 \cdot 1 \cdot (-2)}}{2 \cdot 1}$$

$$x_{1,2} = \frac{-1 \pm \sqrt{1+8}}{2}$$

$$x_{1,2} = \frac{-1 \pm 3}{2} \begin{cases} 1 & (x-1) \\ -2 & (x-(-2)) \end{cases}$$

POZNÁMKA:

NAJDI KOŘENY

$$(x^2 + x - 2)$$

1. DOSAĎ ZA x TAKOVÉ ČÍSLO, ABY VÝSLEDKEM BYLA NULA (ZKOUŠÍM)

$$x^2 + x - 2 = 0$$

JE TO KOŘEN: 1, TEDY $(x-1)$

2. MÁM 1, ČÍM JI VYNÁSOBIT ABYCH DOSTAL -2?

JE TO KOŘEN: -2, TEDY $(x-(-2))$

VÝSLEDEK: $(x-1)(x+2)$

3. KONTROLA

$$-1+2=1$$

$$(x^2 + 1x - 2)$$

$$\int \operatorname{tg}^2 x\,dx = \int \frac{\sin^2 x}{\cos^2 x}\,dx = \int \frac{1-\cos^2 x}{\cos^2 x}\,dx =$$

$$= \int \frac{1}{\cos^2 x}\,dx - \int \frac{\cos^2 x}{\cos^2 x}\,dx =$$

$$= \int \frac{1}{\cos^2 x}\,dx - \int 1\,dx =$$

$$= \underline{\underline{\operatorname{tg} x - x + C}}$$

$$\int \frac{\cos 2x}{\cos x - \sin x}\,dx = \int \frac{\cos^2 x - \sin^2 x}{\cos x - \sin x}\,dx =$$

$$= \int \frac{(\cos x + \sin x)(\cos x - \sin x)}{\cos x - \sin x}\,dx =$$

$$= \int \cos x + \sin x\,dx =$$

$$= \underline{\underline{\sin x - \cos x + C}}$$

$$\int \frac{x^2}{x^2+1} = \int \frac{x^2+1-1}{x^2+1} = \int \frac{x^2+1}{x^2+1} - \frac{1}{x^2+1}\,dx =$$

$$= \int 1 - \frac{1}{x^2+1}\,dx = \underline{\underline{x - \operatorname{arctg} x + C}}$$

NEBO TAKÉ:

$$x + \operatorname{arccotg} x + C$$

POZNÁMKA:

$$\int \frac{x^2}{x^2+1}\,dx$$

VŽDY POKUD:
V ČITATELI JE STEJNÝ STUPEŇ POLYNOMU JAKO VE JMENOVATELI, PROTO POSTUPUJI:

1.) VYDĚLIT POLYNOMY

$$\begin{array}{l} x^2 : (x^2+1) = 1 - \frac{1}{x^2+1} \\ \underline{-(x^2+1)} \\ \quad -1 \end{array}$$

$$\frac{1}{1} - \frac{1}{x^2+1} = \frac{x^2+1-1}{x^2+1}$$

2.) $$\int \frac{x^2+1-1}{x^2+1}\,dx = \int \frac{x^2+1}{x^2+1}\,dx - \int \frac{-1}{x^2+1}\,dx$$

SUBSTITUCE

- NAHRAZENÍ NĚČEHO CO NÁM VADÍ PARAMETREM

$$\int (x^2+5)^3 \cdot 2x\,dx = \left| \begin{aligned} x^2+5 &= s \\ 2x\,dx &= ds \\ dx &= \frac{ds}{2x} \end{aligned} \right| =$$

PROVEDL JSEM DVĚ DERIVACE
1. DERIVACE s JE ds
2. DERIVACE x^2+5 JE $2x$

PAK JSEM ŘEŠIL ROVNICI SE SNAHOU ZÍSKAT dx.

$$= \int s^3 \cdot 2x \cdot \frac{ds}{2x} = \int s^3 \cdot 1\,ds = \int s^3\,ds =$$

$$= \frac{s^4}{4} + C = \frac{(x^2+5)^4}{4} + C$$

$$\int x \sin(3x^2+5)\,dx = \left| \begin{aligned} 3x^2+5 &= z \\ 6x\,dx &= dz \\ dx &= \frac{dz}{6x} \end{aligned} \right| =$$

$$= \int \cancel{x} \sin z \,\frac{dz}{6\cancel{x}} =$$

$$= \int \sin z \,\frac{dz}{6} =$$

$$= \frac{1}{6}\int \sin z\,dz =$$

$$= -\frac{1}{6}\cos z + C =$$

$$= -\frac{1}{6}\cos(3x^2+5) + C$$

$$\int \sqrt{x+10}\,dx = \left|\begin{array}{r} x+10 = s \\ 1dx = ds \\ dx = ds \end{array}\right| =$$

$$= \int \sqrt{s}\,ds = \int s^{\frac{1}{2}}\,ds = \frac{s^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C =$$

$$= \frac{s^{\frac{3}{2}}}{\frac{3}{2}} + C = s^{\frac{3}{2}} \cdot \frac{2}{3} + C =$$

$$= \frac{2s^{\frac{3}{2}}}{3} + C = \underline{\underline{\frac{2(x+10)^{\frac{3}{2}}}{3} + C}}$$

$$\int \frac{\ln x}{x}\,dx = \left|\begin{array}{r} \ln x = s \\ \frac{1}{x}dx = ds \\ dx = x\,ds \end{array}\right| = \int \frac{s}{x}\,x\,ds = \int s\,ds =$$

$$= \frac{s^2}{2} + C = \underline{\underline{\frac{(\ln x)^2}{2} + C}}$$

$$\int \sin^3 x \cdot \cos x\,dx = \int \sin x \cdot \sin^2 x \cdot \cos x\,dx =$$

$$= \left|\begin{array}{r} \sin^2 x = s \\ 2\sin x \cos x\,dx = ds \\ dx = \frac{ds}{2\sin x \cos x} \end{array}\right| =$$

POZNÁMKY

$(\sin^2 x)' = 2\sin x \cdot (\sin)' =$

$= 2\sin x \cdot \cos x$

$$= \int \cancel{\sin x} \cdot s \cancel{\cos x} \frac{ds}{2\cancel{\sin x}\cancel{\cos x}} = \int \frac{s}{2}\,ds = \frac{s^2}{2\cdot 2} + C =$$

$$= \underline{\underline{\frac{(\sin^2 x)^2}{4} + C}}$$

POKUD MÁM SINUS NEBO COSINUS V LICHÉ MOCNINĚ, TAK SE POUŽÍVÁ SUBSTITUCE, POKUD V SUDÉ MOCNINĚ, TAK SE POUŽÍVÁ PER-PARTES

I TAKTO TO JDE:

$$\int \sin^3 x \cdot \cos x \, dx = \left| \begin{array}{rl} \sin x &= t \\ \cos x \, dx &= dt \\ dx &= \frac{dt}{\cos x} \end{array} \right| = \int t^3 \cdot \cos x \frac{dt}{\cos x} =$$

$$= \int t^3 \, dt = \frac{t^4}{4} + C = \underline{\underline{\frac{\sin^4 x}{4} + C}}$$

$$\int \cos^5 x \, dx = \int \cos x \cos^2 x \cos^2 x \, dx = \int \cos x (1-\sin^2 x)(1-\sin^2 x) \, dx =$$

$$= \left| \begin{array}{rl} \sin x &= t \\ \cos x \, dx &= dt \\ dx &= \frac{dt}{\cos x} \end{array} \right| = \int \cos x (1-t^2)(1-t^2) \frac{dt}{\cos x} =$$

$$= \int (1-t^2)(1-t^2) \, dt = \int 1 - t^2 - t^2 + t^4 \, dt =$$

$$= \int 1 - 2t^2 + t^4 \, dt = t - 2\frac{t^3}{3} + \frac{t^5}{5} + C =$$

$$= \underline{\underline{\sin x - 2\frac{\sin^3 x}{3} + \frac{\sin^5 x}{5} + C}}$$

$$\int \frac{1}{(x-3)^2+1}\,dx = \left| \begin{aligned} x-3 &= t \\ 1\,dx &= dt \\ dx &= dt \end{aligned} \right| =$$

$$= \int \frac{1}{t^2+1}\,dt = \operatorname{arctg} t + C =$$

$$= \underline{\underline{\operatorname{arctg}(x-3) + C}}$$

$$\int \frac{1}{x^2+16}\,dx = \int \frac{1}{16\left(\frac{x^2}{16}+1\right)}\,dx = \frac{1}{16}\int \frac{1}{\left(\frac{x}{4}\right)^2+1}\,dx =$$

$$= \left| \begin{aligned} \frac{x}{4} &= t \\ \frac{1}{4}dx &= dt \\ dx &= \frac{dt}{\frac{1}{4}} = 4\,dt \end{aligned} \right| = \frac{1}{16}\int \frac{1}{(t^2+1)} \cdot 4\,dt =$$

$$\left(x \cdot \frac{1}{4}\right)' = (x)' \cdot \frac{1}{4} + x \cdot \left(\frac{1}{4}\right)'$$

$$= \frac{1}{4}\int \frac{1}{t^2+1}\,dt = \frac{1}{4}\operatorname{arctg} t + C =$$

$$= \underline{\underline{\frac{1}{4}\operatorname{arctg}\left(\frac{x}{4}\right) + C}}$$

$$\int \frac{1}{x^2+4x+5}\,dx =$$ PROVEDU ÚPRAVU NA ČTVEREC:

PROTOŽE DISCRIMINANT BY VYŠEL ZÁPORNÝ

$$x^2+4x+5$$

$$(x^2+4x) \quad +5$$

$$\left(x^2+4x+\left(\frac{4}{2}\right)^2\right)-\left(\frac{4}{2}\right)^2+5$$

$$(x^2+4x+2^2)-4+5$$

$$(x+2)^2+1$$

$$= \int \frac{1}{(x+2)^2+1}\,dx = \left| \begin{array}{l} x+2 = s \\ 1\,dx = ds \\ dx = ds \end{array} \right| =$$

$$= \int \frac{1}{s^2+1}\,ds = \operatorname{arctg} s + C =$$

$$= \underline{\underline{\operatorname{arctg}(x+2)+C}}$$

$$\int \frac{1}{x^2+2x+7}\,dx =$$ PROVEDU ÚPRAVU NA ČTVEREC

$$x^2+2x+7$$
$$(x^2+2x) \qquad +7$$
$$\left(x^2+2x+\left(\frac{2}{2}\right)^2\right)-\left(\frac{2}{2}\right)^2+7$$
$$\left(x^2+2x+1^2\right)-1+7$$
$$(x+1)^2+6$$

$$=\int \frac{1}{(x+1)^2+6}\,dx = \int \frac{1}{6\left(\frac{(x+1)^2}{6}+1\right)}\,dx =$$

$$= \frac{1}{6}\int \frac{1}{\left(\frac{x+1}{\sqrt{6}}\right)^2+1}\,dx = \left| \begin{array}{l} \frac{x+1}{\sqrt{6}} = t \\ \frac{1}{\sqrt{6}}\,dx = dt \\ dx = \frac{dt}{\frac{1}{\sqrt{6}}} = \sqrt{6}\,dt \end{array} \right| =$$

$$\left(\frac{x+1}{\sqrt{6}}\right)' = \left((x+1)\cdot\frac{1}{\sqrt{6}}\right)' =$$
$$=(x+1)'\cdot\frac{1}{\sqrt{6}}+(x+1)\cdot\left(\frac{1}{\sqrt{6}}\right)' =$$
$$=\frac{1}{\sqrt{6}}$$

$$= \frac{1}{6}\int \frac{1}{(t^2+1)}\sqrt{6}\,dt = \frac{\sqrt{6}}{6}\int \frac{1}{t^2+1} =$$

$$= \frac{\sqrt{6}}{6}\operatorname{arctg} t + C = \underline{\underline{\frac{\sqrt{6}}{6}\operatorname{arctg} \frac{x+1}{\sqrt{6}} + C}}$$

$$\int \frac{1}{x^2+8x+20}\,dx = \int \frac{1}{(x+4)^2+4}\,dx = \left| \begin{aligned} x+4 &= t \\ 1\,dx &= dt \\ dx &= dt \end{aligned} \right| =$$

PROVEDU ÚPRAVU NA ČTVEREC, protože discriminant je menší než nula.

$$x^2+8x+20$$

$$(x^2+8x) \qquad +20$$

$$\left(x^2+8x+\left(\frac{8}{2}\right)^2\right)-\left(\frac{8}{2}\right)^2+20$$

$$\left(x^2+8x+4^2\right)-4^2+20$$

$$(x+4)^2+4$$

$$= \int \frac{1}{t^2+4}\,dt = \int \frac{1}{4\left(\frac{t^2}{4}+1\right)}\,dt =$$

$$= \frac{1}{4}\int \frac{1}{\left(\frac{t}{2}\right)^2+1} = \left| \begin{aligned} \frac{t}{2} &= s \\ \frac{1}{2}\,dt &= ds \\ dt &= \frac{ds}{\frac{1}{2}} = 2ds \end{aligned} \right| =$$

$$\left(\frac{t}{2}\right)' = \left(t \cdot \frac{1}{2}\right)' =$$

$$= (t)' \cdot \frac{1}{2} + t \cdot \left(\frac{1}{2}\right)' =$$

$$= \frac{1}{2}$$

$$= \frac{1}{4}\int \frac{1}{s^2+1}\,2ds = \frac{2}{4}\int \frac{1}{s^2+1}\,ds =$$

$$= \frac{1}{2}\operatorname{arctg} s + C = \frac{1}{2}\operatorname{arctg}\frac{t}{2} + C =$$

$$= \frac{1}{2}\operatorname{arctg}\frac{x+4}{2} + C$$

METODA SUBSTITUCE - K PROCVIČENÍ

$$\int (2x-5)^7 dx = \left| \begin{aligned} 2x-5 &= s \\ 2\,dx &= ds \\ dx &= \frac{ds}{2} \end{aligned} \right| = \int s^7 \frac{ds}{2} =$$

$$= \frac{1}{2}\int s^7 ds = \frac{1}{2}\,\frac{s^8}{8} + C =$$

$$= \underline{\underline{\frac{(2x-5)^8}{16} + C}} \qquad \text{pro } x \in R$$

KONTROLA:

$$\left(\frac{(2x-5)^8}{16} + C\right)' = \left((2x-5)^8\right)' \cdot \frac{1}{16} + (2x-5)^8 \cdot \left(\frac{1}{16}\right)' + 0 =$$

$$= 8(2x-5)^7 (2x-5)' \cdot \frac{1}{16} + 0 + 0 =$$

$$= 8(2x-5)^7 \cdot 2 \cdot \frac{1}{16} =$$

$$= \frac{8 \cdot 2 \cdot 1}{16}(2x-5)^7 =$$

$$= \underline{(2x-5)^7}$$

$$\int \sin 2x \, dx = \left| \begin{array}{rl} 2x &= t \\ 2dx &= dt \\ dx &= \frac{dt}{2} \end{array} \right| = \int \sin t \cdot \frac{dt}{2} =$$

$$= \frac{1}{2} \int \sin t \, dt = \frac{1}{2} \cdot (-\cos t) + C =$$

$$= \underline{\underline{-\frac{1}{2} \cos 2x + C}} \qquad \text{pro } x \in R$$

JINAK:

$$\int \sin 2x \, dx = \int 2 \sin x \cos x \, dx = 2 \int \sin x \cos x \, dx =$$

$$= \left| \begin{array}{rl} \sin x &= t \\ \cos x \, dx &= dt \\ dx &= \frac{dt}{\cos x} \end{array} \right| = 2 \int t \cos x \frac{dt}{\cos x} = 2 \int t \, dt =$$

$$= 2 \frac{t^2}{2} + C = t^2 + C = \sin^2 x + C =$$

$$= \frac{1 - \cos 2x}{2} + C = \frac{1}{2} - \frac{\cos 2x}{2} + C = \frac{1}{2} - \frac{\cos 2x}{2} - \frac{1}{2} + C =$$

$$= -\frac{\cos 2x}{2} + C = \underline{\underline{-\frac{1}{2} \cos 2x + C}}$$

$$x \in R$$

PŘIDAL JSEM SI V RÁMCI JAKÉKOLIV KONSTANTY C.

GONIOMETRICKÉ VZTAHY:

$$\sin 2x = 2 \sin x \cos x$$

$$\sin^2 x = \frac{1 - \cos 2x}{2} = \frac{1}{1 + \cot g^2 x}$$

$$\int (5x + \sin 7x)\,dx = \int 5x\,dx + \int \sin 7x\,dx = \left| \begin{array}{l} 7x = s \\ 7dx = ds \\ dx = \frac{ds}{7} \end{array} \right| =$$

$$= 5\int x\,dx + \int \sin s\,\frac{ds}{7} =$$

$$= 5\int x\,dx + \frac{1}{7}\int \sin s\,ds =$$

$$= 5 \cdot \frac{x^2}{2} + \frac{1}{7}(-\cos s) + C =$$

$$= \frac{5}{2}x^2 - \frac{1}{7}\cos 7x + C$$

pro $x \in R$

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